

Boundedness in generalized Šerstnev PN spaces

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Abstract

The motivation of this paper is a suggestion by Höle of comparing the notions of \mathcal{D} -boundedness and boundedness in Probabilistic Normed spaces (briefly PN spaces), with non necessarily continuous triangle functions. Such spaces are here called “pre-PN spaces”. Some results on Šerstnev spaces due to B. Lafuerza, J. A. Rodríguez, and C. Sempi, are here extended to *generalized Šerstnev spaces* (these are pre-PN spaces satisfying a more general Šerstnev condition).

We also prove some facts on PN spaces (with continuous triangle functions). First, a connection between fuzzy normed spaces defined by Felbin and certain Šerstnev PN spaces is established. We further observe that topological vector PN spaces are F -normable and paranormable, and also that locally convex topological vector PN spaces are bornological. This last fact allows to describe continuous linear operators between certain generalized Šerstnev spaces in terms of bounded subsets.

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1 Motivation and main results

After the work done by Lafuerza, Rodríguez, and Sempi in [14], Höhle suggested the following:

Problem 1.1 Compare \mathcal{D} -boundedness and boundedness in the PN spaces (V, ν, τ, τ^*) , with τ and τ^* non necessarily continuous.

Such spaces are here called *pre-PN spaces* and include the PN spaces introduced in [3] (where the triangle functions τ and τ^* are assumed to be continuous). Every PN space endows a topology, usually called the *strong topology*, which is always metrizable (thanks to the continuity of τ and τ^*). The probabilistic norm ν is a continuous map from V to the space Δ^+ of distance distribution functions (the last endowed with the Levy-Sybley metric). Recall that a base of strong neighborhoods at $q \in V$ is $\{N_q(t)\}_{t>0}$ where

$$N_q(t) := \{p : \nu_{p-q}(t) > 1 - t\} = q + N_\theta(t),$$

θ being the origin of V .

In the case of pre-PN spaces, we do always have a topology, but the family $\{N_q(t)\}$ still generates a certain “generalized topology” as considered by Fréchet (see Section 2), where a notion of boundedness is possible (see Definition 2.7), as well as the notion of \mathcal{D} -boundedness is obviously extended (see Definition 2.2).

In [14] the authors show that if the PN space is a Šerstnev space which is topologically vectorial, then bounded and \mathcal{D} -bounded subsets coincide. We here extend this result to a class of pre-PN spaces which we call “generalized Šerstnev spaces”, or more precisely ϕ -Šerstnev spaces, where ϕ is a given map \mathbb{R} to \mathbb{R} satisfying certain properties (see Theorem 8.3, and preparatory sections 5 and 7). The well known α -simple spaces are indeed ϕ -Šerstnev PN spaces where $\phi(x) = x^{1/\alpha}$ and $\alpha > 0$. This yields a new interpretation of some results in [13]; see Section 6.

Sections 3 and 4 contain some connections between certain PN spaces with other known structures. For instance, we show that locally convex topologically vector PN spaces are bornological (Proposition 3.4). This allows to determine continuous linear operators between such PN spaces in terms of bounded subsets (Corollary 3.7). This finishes various results and counterexamples in [8] and [14] about linear operators between α -simple spaces. In Theorem 4.5 we observe that topologically vector PN spaces are F -normable and paranormable.

We conclude the paper by comparing \mathcal{D} -bounded and bounded subsets in F -normable spaces (see Section 9).

2 Some preliminaries

2.1 PM spaces and PN spaces

We next recall the definition of PN space given in [3]. However, we here do not assume that the triangle functions involved are continuous. It is convenient to us to consider also “triangle functions” which are non necessarily associative.

As usual, Δ^+ denotes the set of distance distribution functions (briefly, a d.d.f.), i.e. distribution functions with $F(0) = 0$, endowed with the metric topology given by the modified Levy-Sybley metric d_S (see 4.2 in [20]). Let \mathcal{D}^+ consist of those $F \in \Delta^+$ such that $\lim_{x \rightarrow +\infty} F(x) = 1$. Given a real number a , ε_a denotes the distribution function defined as $\varepsilon_a(x) = 0$ if $x \leq a$ and $\varepsilon_a(x) = 1$ if $x > a$. Hence, \mathbb{R}^+ can be viewed as a subspace of Δ^+ . A triangle function τ is a map from $\Delta^+ \times \Delta^+$ into Δ^+ which is commutative, associative, nondecreasing in each variable and has ε_0 as the identity. If τ is non associative we say that it is a *non associative triangle function*.

Recall that a *probabilistic metric space* (briefly, a PM space) is a triple (S, F, τ) where S is a non-empty set, F is a map from $S \times S$ into Δ^+ , called the probabilistic metric, and τ is an associative triangle function, such that:

$$(M1) \quad F_{p,q} = \varepsilon_0 \text{ if and only if } p = q.$$

$$(M2) \quad F_{p,q} = F_{q,p}.$$

$$(M3) \quad F_{p,q} \geq \tau(F_{p,r}, F_{r,q}).$$

When only (M1) and (M2) are required, it is called a *probabilistic semi-metric space* (briefly, PSM space).

A *PN space* (respectively, a *pre-PN space*) is a quadruple (V, ν, τ, τ^*) in which V is a vector space over the field \mathbb{R} of real numbers, the *probabilistic norm* ν is a mapping from V into Δ^+ , τ and τ^* are (respectively, neither necessarily commutative nor associative) triangle functions such that the following conditions are satisfied for all p, q in V (we use ν_p instead of $\nu(p)$):

$$(N1) \quad \nu_p = \varepsilon_0 \text{ if and only if } p = \theta, \text{ where } \theta \text{ denotes the null vector in } V.$$

$$(N2) \quad \nu_{-p} = \nu_p.$$

$$(N3) \quad \nu_{p+q} \geq \tau(\nu_p, \nu_q).$$

$$(N4) \quad \nu_p \leq \tau^*(\nu_{\lambda p}, \nu_{(1-\lambda)p}) \text{ for every } \lambda \in [0, 1].$$

If, instead of (N1), we only have $\nu_\theta = \varepsilon_0$, then we shall speak of a *(pre-) probabilistic pseudo normed space*, briefly a (pre-) PPN space.

If (V, ν, τ, τ^*) be a PN space (with τ non necessarily continuous), then (V, F, τ) is a probabilistic semi-metric space, where $F_{p,q} = \nu_{p-q}$.

The following partial order relation is analogous to the corresponding one for PM spaces (see Section 8.7 of [20]):

Definition 2.1 A pre-PN space $(V, \nu, \tau_1, \tau_1^*)$ is *better* than another pre-PN space $(V, \nu, \tau_2, \tau_2^*)$, with the same V and ν , if the following conditions hold for all $p, q \in V$ and $\lambda \in [0, 1]$:

- $\tau_1(\nu_p, \nu_q) \geq \tau_2(\nu_p, \nu_q)$;
- $\tau_1^*(\nu_{\lambda p}, \nu_{(1-\lambda)p}) \leq \tau_2^*(\nu_{\lambda p}, \nu_{(1-\lambda)p})$.

We do not know if every PN space (V, ν, τ, τ^*) admits a *best-possible* PN structure, in the sense that is better than any other PN space (V, ν, τ', τ'^*) . It would be interesting to study this problem for Menger PN spaces (cf. Section 8.7 in [20]).

The definition of \mathcal{D} -boundedness, which is merely probabilistic and the same as for PM spaces, was introduced in [12]. It obviously extends to pre-PN spaces:

Definition 2.2 A subset $A \subseteq V$ in a pre-PN space (V, ν, τ, τ^*) is called \mathcal{D} -*bounded* if $\lim_{x \rightarrow \infty} R_A(x) = 1$, where R_A is the probabilistic radius of A given by

$$R_A(x) := l^- \inf \{ \nu_p(x) : p \in A \}. \quad (1)$$

2.2 Examples of PN spaces

Recall that a map $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a *t-norm* if it is commutative, associative, nondecreasing in each variable, and has 1 as identity. Then, τ_T is defined as $\tau_T(F, G)(x) := \sup \{ T(F(s), G(t)) : s + t = x \}$, and $\tau_{T^*}(F, G)(x) := \inf \{ T^*(F(s), G(t)) : s + t = x \}$, where T^* is the dual *t-conorm* given by $T^*(x, y) := 1 - T(1 - x, 1 - y)$. Notice that if T is left-continuous then τ_T is a triangle function [20, p. 100], although it is not necessary. For instance, if Z is the minimum *t-norm*, defined as $Z(x, 1) = Z(1, x) = x$ and $Z(x, y) = 0$, elsewhere, then τ_T is a triangle function.

A *Menger PM space* under a *t-norm* T is a PM space of the form (V, ν, τ_T) . A *Menger PN space* (respectively, Menger pre-PN) under T is a PN space (respectively, pre-PN space) of the form $(V, \nu, \tau_T, \tau_{T^*})$.

A *Šerstnev (pre-) PN space* is a (pre-) PN space (V, ν, τ, τ^*) where ν satisfies the following Šerstnev condition:

$$(\check{S}) \quad \nu_{\lambda p}(x) = \nu_p\left(\frac{x}{|\lambda|}\right), \text{ for all } x \in \mathbb{R}^+, p \in V \text{ and } \lambda \in \mathbb{R} \setminus \{0\}.$$

It turns out that (\check{S}) is equivalent to have (N2) and

$$\nu_p = \tau_M(\nu_{\lambda p}, \nu_{(1-\lambda)p}), \quad (2)$$

for all $p \in V$ and $\lambda \in [0, 1]$ (see [3, Theorem 1]), where M is the *t-norm* defined as $M(x, y) = \min\{x, y\}$. Therefore, condition (N4) is satisfied for every τ^* such that $\tau_M \leq \tau^*$.

In the sense of Definition 2.1, if (V, ν, τ, τ^*) is a Šerstnev PN space then (V, ν, τ, τ_M) is a “better” structure than (V, ν, τ, τ^*) .

Example 2.3 Every normed space $(V, \|\cdot\|)$ yields a Šerstnev, Menger space (V, ν, τ_M, τ_M) , where $\nu_p = \varepsilon_{\|p\|}$. (Recall that $\tau_{M^*} = \tau_M$).

More generally, when the image of ν lies in $\mathbb{R}^+ \subset \Delta^+$, i.e $\nu_p = \varepsilon_{g(p)}$ for some function $g : V \rightarrow \mathbb{R}^+$, we obtain in Section 4 the link with F -norms.

Example 2.4 Let $(V, \|\cdot\|)$ be a normed space, $G \in \Delta^+$ be different from ε_0 and ε_∞ , and $\alpha \geq 0$. Define $\nu : V \rightarrow \Delta^+$ by $\nu_\theta = \varepsilon_0$ and for $p \neq \theta$:

$$\nu_p(t) := G\left(\frac{t}{\|p\|^\alpha}\right). \quad (3)$$

The triple $(V, G; \alpha)$ is called the α -simple space generated by $(V, \|\cdot\|)$ and G . For $\alpha = 1$ we have that (V, ν, τ_M, τ_M) is a Menger PN space, that is Šerstnev (see Theorem 2.1 of [13]). If $G = \varepsilon_1$ we recover Example 2.3. For $\alpha \neq 1$ we do not have a Menger PN space in general, but a PN space with $\tau = \tau^* = \tau_{M,L}$, for some L (see Proposition 6.1).

2.3 Fuzzy normed spaces

The class of PN spaces has some connection with the class of fuzzy normed spaces. We want here establish this connection without giving many details. This yields a source of examples going in both directions; see a similar connection for PM spaces and fuzzy metric spaces in [18, Section 4]. The first definition of fuzzy norm was given by Katsaras [9], and later extended by Felbin [6]. However, as far as we know, there is only the article by Wu and Ma [23] relating fuzzy norms and probabilistic norms as defined in their origin. Recall from [6] that a fuzzy normed space is a quadruple $(V, \|\cdot\|, L, R)$ where V is a real vector space, $\|\cdot\|$ is a function from V to the set of fuzzy numbers, L and R are a continuous t -norm and a t -conorm satisfying certain properties.

Proposition 2.5 *If $(V, \|\cdot\|, L, R)$ is a fuzzy normed space, with L and R continuous t -norm and t -conorm, respectively, then $(V, \nu, \tau_{L^*}, \tau_M)$ is a Šerstnev PN space such that*

$$\nu_{p+q} \leq \tau_{L^*}(\nu_p, \nu_q),$$

for all p and q in S , where ν_p is the distribution function associated to the fuzzy number $\|p\|$.

Conversely we have:

Proposition 2.6 *If (V, ν, τ_T, τ_M) is a Šerstnev PN space and furthermore,*

$$\nu_{p+q} \leq \tau_{L^*}(\nu_p, \nu_q),$$

for all p and q in S , then $(V, \|\cdot\|, L, T^)$ is a fuzzy normed space.*

2.4 The generalized strong topology

Recall from [20, Section 12] (see also [7]) that every PSM space (S, F, τ) endows a generalized topology of type V_D (in the sense of Fréchet), which is Fréchet-separated and first-numerable. It is called the *generalized strong topology*. The associated *strong neighborhood system* is given by $\mathcal{N} = \bigcup_{p \in S} \mathcal{N}_p$, where $\mathcal{N}_p = \{N_p(t) : t > 0\}$ and

$$N_p(t) := \{q \in S : F_{p,q}(t) > 1 - t\}.$$

A countable base at p is given by $\{N_p(1/n) : n \in \mathbb{N}\}$. If we define $\delta(p, q) := d_S(F_{p,q}, \varepsilon_0)$, then δ is a semi-metric on S , and the neighborhood $N_p(t)$ is precisely the open ball $\{q : d_S(F_{p,q}, \varepsilon_0) < t\}$.

If (S, F, τ) is a PM space with τ continuous, the generalized strong topology is a genuine topology called the *strong topology*. Because of (M1) (see subsection 2.1) the strong topology is Hausdorff-separated. Since it is first-numerable and uniformable, one has that it is metrizable.

For a pre-PN space (V, ν, τ, τ^*) we have $N_p(t) = p + N_\theta(t)$, i.e. the generalized topology is invariant under translations. The base of θ -neighborhoods $\{N_\theta(1/n) : n \in \mathbb{N}\}$ determines completely the associated generalized topology. This is also Fréchet-separated, and countably generated by radial and circled θ -neighborhoods. In fact a converse result also holds; see [17] for more details.

According to this setting, we give the following definition:

Definition 2.7 A subset $A \subset V$ in a pre-PN space (V, ν, τ, τ^*) is *bounded* if for every integer $m \geq 1$, there is a finite set $A_1 \subseteq A$ and a natural number $k \geq 1$ such that

$$A \subseteq \bigcup_{p \in A_1} (p + N_\theta(1/m)^{[k]}). \quad (4)$$

where here $B^{[k]} = B + \dots + B$.

2.5 TV groups and TV spaces

Recall that a vector space endowed with a topology, is a *topological vector space* (briefly, a TV space) if both the addition $+: V \times V \rightarrow V$ and multiplication by scalars $\eta : \mathbb{R} \times V \rightarrow V$

are continuous. If only the addition is assumed to be continuous then V is a *topological group*; if furthermore η is continuous at the second place, then it is called a *topological vector group* (briefly, a TV group). In [4] the authors showed that PN spaces with τ continuous, are topological vector groups. We quote this result for further reference.

Theorem 2.8 ([4]) *A PN space (V, ν, τ, τ^*) , with τ continuous, is a TV space if and only if the map η is continuous at the first place (i.e. for every fixed $p \in V$, $\lambda_n p \rightarrow 0$ whenever $\lambda_n \rightarrow 0$). \square*

The following conditions to have a TV space are sufficient (see Theorem 4 and remarks after Theorem 5 in [4]): $\nu_p \neq \epsilon_\infty$, for all $p \in V$, the subset $\nu(V)$ is closed in (Δ^+, d_L) , τ^* is continuous, and τ^* Archimedean on $\nu(V)$.

For Šerstnev PN spaces Theorem 2.8 yields the following characterization:

Theorem 2.9 ([16]) *A Šerstnev PN space (V, ν, τ, τ^*) is a TV space if and only if $\nu(V) \subseteq \mathcal{D}^+$. \square*

If a PN space (V, ν, τ, τ^*) is a TV space then then a subset is bounded (in the sense of 2.7 if and only if for every integer $m \geq 1$, there is a natural number $k \geq 1$ such that

$$A \subseteq kN_\theta(1/m). \quad (5)$$

This is also equivalent to being “topologically bounded” (as defined in [3]), that is, for every sequence $(\alpha_n) \subset \mathbb{R}$ with $\lim_n \alpha_n = 0$, and for every sequence $(p_n) \subset A$, then $\lim \alpha_n p_n = \theta$ in the strong topology.

3 Normable and bornological PN spaces

Normability of PN spaces has been recently studied in [16]. The following criterium establishes when a PN space is normable (see [21, p. 41]):

Theorem 3.1 (Kolmogorov) *A TV PN space (V, ν, τ, τ^*) is normable if, and only if, there exists a bounded and convex θ -neighborhood. \square*

By a Prochaska’s result adapted to the theory of PN spaces (see [16]) we have that all Šerstnev and Menger PN spaces (V, ν, τ_M, τ_M) are locally convex.

Example 3.2 ([16]) Let $(V, \|\cdot\|; G)$ be a simple space and (V, ν, τ_M, τ_M) the associated Šerstnev and Menger PN space of Example 2.4. If $G \in \mathcal{D}^+$ and strictly increasing then the strong topology is $\|\cdot\|$ -normable.

Remark 3.3 In the previous example, if $G \notin \mathcal{D}^+$ then the associated strong topology is discrete, therefore it is not a TV space (it is just a TV group).

A locally convex TV space E is *bornological* if every circled, convex subset $A \subset E$ that absorbs every bounded set in E is a neighborhood of θ . It is known that metrizable and locally convex topological vector spaces are bornological (see [21, II 8.1]). Bornological spaces are inductive limits of normable spaces ([21, II 8.4]).

Proposition 3.4 *Every PN spaces (V, ν, τ, τ^*) that is a locally convex TV space is bornological.* \square

In Proposition 6.1 we will see that α -simple spaces are PN spaces.

Example 3.5 Let $L(x, y) = (x^{1/\alpha} + y^{1/\alpha})^\alpha$. Then the α -simple PN space $(V, \nu, \tau_{M,L}, \tau_{M,L})$ where $\nu_p(x) = G(x/\|p\|^\alpha)$, with $G \in \mathcal{D}^+$ is bornological.

A linear operator $T : V_1 \rightarrow V_2$ is called *bounded* if it transforms bounded subsets of V_1 into bounded subsets of V_2 (see e.g. [5, p. 63]). Obviously, continuous linear operators are bounded, but not conversely. However, if the source space is bornological and the target is a locally convex TV space then the converse holds (see e.g. [5, p. 477]). In particular, we have:

Theorem 3.6 *A linear operator between two locally convex TV PN spaces is continuous if, and only if, it is bounded.* \square

Example 3.5 in [14] gives a bounded linear operator from a non bornological (non locally convex) PN space which is not continuous.

Corollary 3.7 *Let G and G' be in \mathcal{D}^+ . Let (V, G, α) and (V, G', α') be two α -simple spaces regarded as PN spaces. A linear operator $T : (V, G, \alpha) \rightarrow (V, G', \alpha')$ is continuous if and only if T is bounded.*

This corollary closes the results in [8] and [14, Section 3].

4 F -normable and paranormable PN spaces

Recall from [21] and [22, Section 4] that an F -norm on a vector space V is a map $g : V \rightarrow \mathbb{R}^+$ such that

- (i) $g(p) = 0$ if and only if $p = \theta$.

(ii) $g(\lambda p) \leq g(p)$ if $|\lambda| \leq 1$.

(iii) $g(p + q) \leq g(p) + g(q)$.

The pair (V, g) is called an *F-normed space*. It is a TV group with respect to the metric $d(p, q) = g(p - q)$, but in general it is not a TV space. *F*-normed spaces which are TV spaces are called *paranormed spaces* (see [22, Section 4]).

Example 4.1 Let V be the vector space of all continuous functions $p : \mathbb{R} \rightarrow \mathbb{R}$, $g(p) := \sup_{t \in \mathbb{R}} \frac{|p(x)|}{a + |p(x)|}$, with $a > 0$. Then g is an *F*-norm but not a paranorm (see [21, Exercise 12(b), p. 35]).

Of course, different *F*-norms may induce the same metric-topology. For instance, if $(V, \|\cdot\|)$ is a normed space then $g(p) = \|p\|^\alpha$, or $g(p) = \frac{\|p\|}{\alpha + \|p\|}$, where $\alpha > 0$, are *F*-norms which induce the same topology as $\|\cdot\|$. Observe that every *F*-normed (respectively, paranormed) space (V, g) is homeomorphic to an *F*-normed (respectively, paranormed) space (V, g') with $g'(V) < 1$. Indeed, if g is an *F*-norm, then $g'(p) = g(p)/(1 + g(p))$ is an *F*-norm equivalent to g .

The above condition (ii) implies $\|-p\| = \|p\|$. This observation and the fact that $\tau_M(\varepsilon_a, \varepsilon_b) = \varepsilon_{a+b}$ yield easily the following correspondence between *F*-norms and certain PN spaces.

Proposition 4.2 *Let $g : V \rightarrow \mathbb{R}^+$ be any map and define ν by $\nu_p := \varepsilon_{g(p)}$. Then (V, g) is an *F*-normed space if, and only if, $(V, \nu, \tau_M, \mathbf{M})$ is a PN space, where \mathbf{M} is defined as $\mathbf{M}(F, G)(x) = M(F(x), G(x))$. \square*

Notice that \mathbf{M} is the maximal triangle function, so $(V, \varepsilon_g, \tau_M, \mathbf{M})$ could not be the best PN structure for a given *F*-norm g . Indeed, if g is a norm we can replace \mathbf{M} by τ_M .

Proposition 4.3 *Let $g : V \rightarrow \mathbb{R}^+$ be any map and define ν by $\nu_p = \varepsilon_{g(p)}$. Let τ and τ^* be two triangle functions.*

1. *If $\tau(\varepsilon_a, \varepsilon_b) \geq \varepsilon_{a+b}$, for all $a, b \in \mathbb{R}^+$, and (V, ν, τ, τ^*) is a PN space, then g is an *F*-norm.*
2. *If $\tau(\varepsilon_a, \varepsilon_b) \leq \varepsilon_{a+b}$, for all $a, b \in \mathbb{R}^+$, and g is an *F*-norm, then (V, ν, τ, τ^*) is a PN space if and only if (N_4) holds.*
3. *If $\tau(\varepsilon_a, \varepsilon_b) \leq \varepsilon_{a+b}$, for all $a, b \in \mathbb{R}^+$, then g is a norm if and only if (V, ν, τ, τ^*) is a Šerstnev PN space. \square*

Proposition 4.4 *Suppose that $(V, \varepsilon_g, \tau, \tau^*)$ is a PN space, with g an F -norm on V . Then η is continuous at the first place. In this case, the strong topology is equivalent to the metric-topology induced by g .*

PROOF. It is easy to check that the strong neighborhood $N_\theta(t)$ coincides with the open ball $\{p : g(p) < t\}$. \square

Conversely, we have the following theorem for TV PN spaces:

Theorem 4.5 *Let (V, ν, τ, τ^*) be a metrizable PN space that is a TV space, then it is paranormable.*

PROOF. Theorem 6.1 in [21, p. 28] implies that metrizable TV spaces are F -normable, therefore by Theorem 2.8 it is paranormable. \square

5 ϕ -transforms on PN spaces

Following [2], for $0 < b \leq \infty$, let M_b be the set of m -transforms which consists on all continuous and strictly increasing functions from $[0, b]$ onto $[0, \infty]$. More generally, let \widetilde{M} be the set of non decreasing left-continuous functions $\phi : [0, \infty] \rightarrow [0, \infty]$ with $\phi(0) = 0$, $\phi(\infty) = \infty$ and $\phi(x) > 0$, for $x > 0$. Then, $M_b \subset \widetilde{M}$ once m is extended to $[0, \infty]$ by $m(x) = \infty$ for all $x \geq b$. Notice that a function $\phi \in \widetilde{M}$ is bijective if and only if $\phi \in M_\infty$.

Sometimes, the probabilistic norms ν and ν' of two given (pre-) PN spaces satisfy $\nu' = \nu\phi$ for some $\phi \in \widetilde{M}$, non necessarily bijective. Let ϕ^\wedge be the (unique) quasi-inverse of ϕ which is left-continuous. Recall from [20, p. 49] that ϕ^\wedge is defined by $\phi^\wedge(0) = 0$, $\phi^\wedge(\infty) = \infty$ and $\phi^\wedge(t) = \sup\{u : \phi(u) < t\}$, for all $0 < t < \infty$. It follows that $\phi^\wedge(\phi(x)) \leq x$ and $\phi(\phi^\wedge(y)) \leq y$ for all x and y .

One has the following (which generalizes Theorem 4 in [2]):

Theorem 5.1 *Let (S, F) be a PPM space, and $F' = F\phi$ with $\phi \in \widetilde{M}$. Then, (S, F') is a PPM space. Moreover, the generalized strong topology induced by F is finer than the one induced by F' . If $\phi^\wedge(y) > 0$, for $y > 0$, then they coincide.*

PROOF. We have that $\phi(x) > 0$, for all $x > 0$. Hence, for each $m \in \mathbb{N}$ there is an $n \in \mathbb{N}$, with $n \geq m$ such that $\phi(1/m) > 1/n$. Thus, for every $p, q \in S$ satisfying $F_{p,q}(1/n) > 1 - n$, we have

$$F_{p,q}(1/m) = F_{p,q}(\phi(1/m)) \geq F_{p,q}(1/n) > 1 - 1/n \geq 1 - 1/m,$$

i.e. every strong neighborhood $N'_p(1/m)$ with respect to F' contains a strong neighborhood $N_p(1/n)$ with respect to F . \square

The following consequences are straightforward:

Corollary 5.2 *Let $V_1 = (V, \nu, \tau, \tau^*)$ and $V_2 = (V, \nu', \tau', (\tau^*)')$ be two pre-PN spaces with the same base vector space and suppose that $\nu' = \nu\phi$, for some $\phi \in \widetilde{M}$. Then the following hold:*

- (i) *If the scalar multiplication $\eta : \mathbb{R} \times V \rightarrow V$ is continuous at the first place with respect to ν , then it is so with respect to ν' . In particular, if τ and τ' are continuous, and V_1 is a TV PN space, then so is V_2 .*
- (ii) *If $\lim_{x \rightarrow \infty} \phi(x) = \infty$ and A is a \mathcal{D} -bounded in V_1 then it so in V_2 .*
- (iii) *If A is bounded in V_1 then it is so in V_2 . \square*

If (V, ν, τ, τ^*) is a given pre-PN space and $\phi \in \widetilde{M}$, we can consider the composite $\nu' := \nu\phi$ from V into Δ^+ . By Theorem 5.1 ν' satisfies (N1) and (N2). We can consider the quadruple $(V, \nu\phi, \tau^\phi, (\tau^*)^\phi)$, where τ^ϕ is given by

$$\tau^\phi(F, G)(x) := \tau(F\phi^\wedge, G\phi^\wedge)\phi(x), \quad (6)$$

and $\tau^{*\phi}$ is defined in a similar way. The quadruple $(V, \nu\phi, \tau^\phi, (\tau^*)^\phi)$ is called the ϕ -transform of (V, ν, τ, τ^*) .

Proposition 5.3 *Let (V, ν, τ, τ^*) be a pre-PN space. If $\phi \in \widetilde{M}$ then the ϕ -transform $(V, \nu\phi, \tau^\phi, (\tau^*)^\phi)$ is a pre-PN space. \square*

Remark 5.4 If $\phi \notin M_\infty$, then associativity of τ^ϕ and $\tau^{*\phi}$ might fail. But, if $\phi \in M_\infty$ then τ^ϕ and $\tau^{*\phi}$ are (associative) triangle functions. Hence, in this case the ϕ -transform of a PN space is a PN space. Notice also that the ϕ^{-1} -transform of $(V, \nu\phi, \tau^\phi, (\tau^*)^\phi)$ is the space (V, ν, τ, τ^*) .

As in [20, 7.1.7] let \mathcal{L} be the set of all binary operations L on $[0, +\infty]$ which are surjective, non decreasing in each place and continuous on $[0, +\infty] \times [0, +\infty]$, except possibly at the points $(0, +\infty)$ and $(+\infty, 0)$. If $\phi \in M_\infty$ and we define $L(x, y) = \phi^{-1}(\phi(x) + \phi(y))$, then $L \in \mathcal{L}$. Given a continuous t -norm T , one can consider the triangle functions $\tau_{T,L}$ and $\tau_{T^*,L}$ which are defined in [20, 7.2]. An easy calculation yields the following result:

Theorem 5.5 *Let $(V, \nu, \tau_T, \tau_{T^*})$ be a Menger PN space under some continuous t -norm T , and $\phi \in M_\infty$. Then, the PN space $(V, \nu\phi, \tau_{T,L}, \tau_{T^*,L})$ is the ϕ -transform of $(V, \nu, \tau_T, \tau_{T^*})$.*

Notice that this is a Menger space under T if, and only if, $\phi(x) = kx$ for some constant $k \in \mathbb{R} \setminus 0$ (cf. [11, Section 6]).

6 α -simple PN spaces

As we have seen in Example 2.4, the way to produce a Menger PN space under M from a simple space $(V, \|\cdot\|, G)$ does not need any assumption on the distribution function G . However, in the case of α -simple spaces, some restrictions on G are required in order to obtain the structure of Menger PN space under a certain t -norm T_G (see Section 3 in [13]). In this section we give a new proof of Theorem 3.1, part (a) of [13], by using the following:

Proposition 6.1 *If (V, G, α) is an α -simple space, and $\nu_p(t) = G(t / \|p\|^\alpha)$, then $(V, \nu, \tau_{M,L}, \tau_{M,L})$ is a PN space, where $L \in \mathcal{L}$ and $L(x, y) = (x^{1/\alpha} + y^{1/\alpha})^\alpha$. \square*

PROOF. This is a particular case of Theorem 5.5, with $\phi(x) = x^{1/\alpha}$. \square

Now, suppose that $G \in \Delta^+$ is strictly increasing. Consider the t -norm T_G defined as follows:

$$T_G(x, y) := G \left(\left\{ [G^{-1}(x)]^{1/(1-\alpha)} + [G^{-1}(y)]^{1/(1-\alpha)} \right\}^{(1-\alpha)} \right).$$

Corollary 6.2 ([13]) *Let (V, G, α) be an α -simple space, where G is an strictly increasing continuous distribution function, and $\alpha > 1$. Then $(V, \nu, \tau_{T_G}, \tau_{T_G}^*)$ is a Menger PN space under T_G .*

PROOF. Let $\tau = \tau_{T_G}$ in the above proposition. We have to see that $(V, \nu, \tau_{M,L}, \tau_{M,L})$ is better than (V, ν, τ, τ^*) in the sense of Definition 2.1. For that, we have to show that $\tau_{M,L}(\nu_p, \nu_q) \geq \tau(\nu_p, \nu_q)$ and $\tau_{M,L}(\nu_{\lambda p}, \nu_{(1-\lambda)p}) \leq \tau^*(\nu_{\lambda p}, \nu_{(1-\lambda)p})$, for all $p, q \in V$ and $\lambda \in (0, 1)$.

$$\begin{aligned} \tau(\nu_p, \nu_q)(x) &= \sup_{r+s=x} \{T_G(\nu_p(r), \nu_q(s))\} = \\ &= \sup_{r+s=x} \left\{ G \left(\left[G^{-1} \left(G \left(\frac{r}{\|p\|^\alpha} \right) \right)^{1/(1-\alpha)} + \left[G^{-1} \left(G \left(\frac{s}{\|q\|^\alpha} \right) \right) \right]^{1/(1-\alpha)} \right]^{(1-\alpha)} \right) \right\} = \\ &= \sup_{r+s=x} \left\{ G \left(\left(\frac{r}{\|p\|^\alpha} \right)^{1/(1-\alpha)} + \left(\frac{s}{\|q\|^\alpha} \right)^{1/(1-\alpha)} \right)^{(1-\alpha)} \right\}. \end{aligned}$$

On the other hand

$$\begin{aligned} \tau_{M,L}(\nu_p, \nu_q)(x) &= \sup_{L(u,v)=x} \{M(\nu_p(u), \nu_q(v))\} = \\ &= \sup_{u^{1/\alpha} + v^{1/\alpha} = x^{1/\alpha}} \left\{ M \left(G \left(\frac{u}{\|p\|^\alpha} \right), G \left(\frac{v}{\|q\|^\alpha} \right) \right) \right\} = \\ &= G \left(\frac{x}{(\|p\| + \|q\|)^\alpha} \right). \end{aligned}$$

Now, we use one of the known Hölder's inequalities

$$(a + b)^{1-\alpha} \leq \lambda^\alpha a^{1-\alpha} + (1 - \lambda)^\alpha b^{1-\alpha},$$

which holds for $\alpha > 1$, $\lambda \in (0, 1)$ and $a, b \in (0, +\infty)$. By setting

$$a := \left(\frac{r}{\|p\|^\alpha} \right)^{1/(1-\alpha)} \quad b := \left(\frac{s}{\|q\|^\alpha} \right)^{1/(1-\alpha)} \quad \lambda := \frac{\|p\|^\alpha}{(\|p\| + \|q\|)^\alpha}$$

it follows

$$\left(\frac{r}{\|p\|^\alpha} \right)^{1/(1-\alpha)} + \left(\frac{s}{\|q\|^\alpha} \right)^{1/(1-\alpha)} \leq \left(\frac{r+s}{\|p\| + \|q\|} \right)^{1/(1-\alpha)}$$

After applying G in both sides, we obtain one of the desired inequalities $\tau_{M,L}(\nu_p, \nu_q) \geq \tau(\nu_p, \nu_q)$. The other inequality follows analogously. \square

A similar result can be shown for $\alpha < 1$ by choosing a t -norm T as in Theorem 3.2, part (a) of [13].

7 ϕ -transforms on Šerstnev spaces

If (V, ν, τ, τ^*) is a Šerstnev pre-PN space and $\nu' := \nu\phi$ for some bijective function $\phi \in M_\infty$ (see the Section 5). Then ν' will satisfy $\nu'_{\lambda p}(x) = \nu'_p \left(\phi^{-1} \left(\frac{\phi(x)}{|\lambda|} \right) \right)$, for all $x \in \mathbb{R}^+$, $p \in V$ and $\lambda \in \mathbb{R} \setminus \{0\}$. This motivates the following definition for ϕ non necessarily bijective.

Definition 7.1 We say that a quadruple (V, ν, τ, τ^*) satisfies the ϕ -Šerstnev condition if:

$$(\phi\text{-}\check{S}) \quad \nu_{\lambda p}(x) = \nu_p \left(\phi^\wedge \left(\frac{\phi(x)}{|\lambda|} \right) \right), \text{ for all } x \in \mathbb{R}^+, p \in V \text{ and } \lambda \in \mathbb{R} \setminus \{0\}.$$

A pre-PN space (V, ν, τ, τ^*) which satisfies the ϕ -Šerstnev condition is called a ϕ -Šerstnev pre-PN space.

Example 7.2 If $\phi(x) = x^{1/\alpha}$, for a fixed positive real number α , then condition $(\phi\text{-}\check{S})$ takes the form:

$$(\alpha\text{-}\check{S}) \quad \nu_{\lambda p}(x) = \nu_p \left(\frac{x}{|\lambda|^\alpha} \right), \text{ for all } x \in \mathbb{R}^+, p \in V \text{ and } \lambda \in \mathbb{R} \setminus \{0\}.$$

Pre-PN spaces satisfying $(\alpha\text{-}\check{S})$ are called α -Šerstnev. (Note 1-Šerstnev is just Šerstnev.) We will see in Proposition 6.1 that α -simple spaces give rise to α -Šerstnev PN spaces of the form $(V, \nu, \tau_{M,L}, \tau_{M,L})$ where $L(x, y) := (x^{1/\alpha} + y^{1/\alpha})^\alpha$. Thus, α -simple pre-PN spaces can be viewed as ϕ -transforms of PN simple spaces.

More generally, the ϕ -transform of a Šerstnev PN space is a ϕ -Šerstnev pre-PN space, if ϕ is bijective. This yields the following characterization for ϕ -Šerstnev pre-PN spaces.

Proposition 7.3 *Let $L(x, y) = \phi^{-1}(\phi(x) + \phi(y))$ with $\phi \in M_\infty$. Then $(\phi\text{-}\check{S})$ holds if and only if (N2) and also*

$$\nu_p = \tau_{M,L}(\nu_{\lambda p}, \nu_{(1-\lambda)p}),$$

for every $p \in V$ and $\lambda \in [0, 1]$ are satisfied. In particular, ϕ -Šerstnev spaces admit a better pre-PN structure of the form $(V, \nu, \tau, \tau_{M,L})$.

PROOF. By [20, Section 7.7], taking quasi-inverses we have that

$$\nu_p = \tau_{M,L}(\nu_{\lambda p}, \nu_{(1-\lambda)p}) \iff \nu_p^\wedge = L(\nu_{\lambda p}^\wedge, \nu_{(1-\lambda)p}^\wedge).$$

By definition of L , this is equivalent to $\phi\nu_p^\wedge = \phi\nu_{\lambda p}^\wedge + \phi\nu_{(1-\lambda)p}^\wedge$. Taking again quasi-inverses, we obtain $\nu_p\phi^{-1} = \tau_M(\nu_{\lambda p}\phi^{-1}, \nu_{(1-\lambda)p}\phi^{-1})$. This condition together with (N2) is equivalent to the Šerstnev condition for $\nu\phi$. \square

For α -Šerstnev spaces one also has a slightly different characterizing formula.

Proposition 7.4 *Let $\alpha \in \mathbb{R}^+$. Then $(\alpha\text{-}\check{S})$ holds if, and only if, (N2) and*

$$\nu_{\beta p} = \tau_M(\nu_{\lambda p}, \nu_{(1-\lambda)p}), \tag{7}$$

for every $p \in V$ and $\lambda \in [0, 1]$ are satisfied, where $\beta = [\lambda^\alpha + (1 - \lambda)^\alpha]^{1/\alpha}$.

PROOF. Suppose first that $(\alpha\text{-}\check{S})$ is satisfied. Then, obviously $\nu_{-p} = \nu_p$, hence (N2) holds. As in the proof of the previous proposition, we have that (7) holds if, and only if,

$$\nu_{\beta p}^\wedge = \nu_{\lambda p}^\wedge + \nu_{(1-\lambda)p}^\wedge, \tag{8}$$

for all $p \in V$ and all $\lambda \in [0, 1]$. Then, because of $(\alpha\text{-}\check{S})$,

$$\nu_{\lambda p}(x) = \nu_p\left(\frac{x}{\lambda^\alpha}\right) \iff \nu_{\lambda p}^\wedge = \lambda^\alpha \nu_p^\wedge,$$

for every $\lambda \in [0, 1]$ and for every $p \in V$, so that (7) holds easily.

Conversely, suppose that (N2) and (7) hold. Let $g : \mathbb{R}^+ \rightarrow \mathbb{R}$ be defined as

$$g(z) := \nu_{zp}^\wedge(t),$$

for fixed $t \in [0, 1]$ and $p \in V$. Then g is a non-decreasing map such that

$$g[(\lambda^\alpha + (1 - \lambda)^\alpha)^{1/\alpha} z] = g(\lambda z) + g[(1 - \lambda)z].$$

Define now $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ by $f(z) := g(z^{1/\alpha})$. Then, f is a nondecreasing function that satisfies $f(x + y) = f(x) + f(y)$, for all x and $y \in \mathbb{R}^+$. This is the Cauchy equation, therefore by [1, Corollary 5] we have $f(x) = f(1) \cdot x$, that is $g(x^{1/\alpha}) = g(1) \cdot x$. By taking $z = x^{1/\alpha}$, we obtain $g(z) = g(1)z^\alpha$, and hence $\nu_{zp}^\wedge(t) = z^\alpha \nu_p^\wedge(t)$. This last equality yields $(\alpha\text{-}\check{S})$, as desired. \square

Corollary 7.5 *If (V, ν, τ, τ^*) is an α -Šerstnev space, then (V, ν, τ, τ_M) is also an α -Šerstnev space.*

PROOF. This follows from the inequalities $\nu_p(x) \leq \nu_{\beta p}(x) = \tau_M(\nu_{\lambda p}, \nu_{(1-\lambda)p})$. \square

8 Boundedness in ϕ -Šerstnev spaces

An example in [14] shows that even for normable PN spaces bounded subsets and D -bounded subsets do not coincide. They do coincide on the Šerstnev PN spaces that are TV spaces ([14, Theorem 2.3]). In fact, for Šerstnev spaces which are not TV spaces, \mathcal{D} -bounded subsets are bounded, but the converse might fail as we illustrate with Example 8.4.

We first generalize the following:

Theorem 8.1 ([16]) *A Šerstnev PN space (V, ν, τ, τ^*) is a TV space if, and only if, $\nu(V) \subseteq \mathcal{D}^+$. \square*

Theorem 8.2 *Let $\phi \in \widetilde{M}$ such that $\lim_{x \rightarrow \infty} \phi^\wedge(x) = \infty$. Let (V, ν, τ, τ^*) be a ϕ -Šerstnev pre-PN space. Then scalar multiplication $\eta : \mathbb{R} \times V \rightarrow V$ is continuous at the first place if and only if $\nu(V) \subseteq \mathcal{D}^+$.*

PROOF. If ν maps V into \mathcal{D}^+ , then, for every $x > 0$ and every sequence $\{\alpha_n\}$ converging to 0, one has:

$$\nu_{\alpha_n p}(x) = \nu_p \left(\phi^\wedge \left(\frac{\phi(x)}{|\alpha_n|} \right) \right) \longrightarrow 1,$$

as n tends to $+\infty$ (we use the fact that $\lim_{y \rightarrow \infty} \phi^\wedge(y) = \infty$), whence the assertion.

Conversely, suppose that η is continuous at the first place. For every $n \geq 1$, let $x_n = \phi^\wedge(n\phi(1))$. Then, for all $p \in V$,

$$\nu_p(x_n) = \nu_p(\phi^\wedge(n\phi(1))) = \nu_p \left(\phi^\wedge \left(\frac{\phi(1)}{1/n} \right) \right) = \nu_{p/n}(1) \longrightarrow 1.$$

The last term converges to 1 by assumption. Therefore, $\nu_p(x) \rightarrow 1$ whenever x tends to infinity, as desired. \square

A remarkable result in [14] is Theorem 2.3, where it is shown that in a Šerstnev space that is a TV space, a subset is \mathcal{D} -bounded if, and only if, it is bounded or “topologically bounded” (fact that it has been observed in the introduction to be same). We extend this result to ϕ -Šerstnev spaces in the following theorem with almost the same proof. Notice that the implicit assumption in [14] that they are TV spaces is not necessary at all for the first part. The restriction to TV spaces generalizes a result in [16].

Theorem 8.3 *Let $\phi \in \widetilde{M}$ such that $\lim_{x \rightarrow \infty} \phi^\wedge(x) = \infty$. Let (V, ν, τ, τ^*) be a ϕ -Šerstnev pre-PN space. Then for a subset $A \subset V$ the following statements are equivalent:*

- (a) *For every $n \in \mathbb{N}$, there is a $k \in \mathbb{N}$ such that $A \subset kN_\theta(1/n)$.*
- (b) *A is \mathcal{D} -bounded.*

These equivalent conditions imply:

- (c) *A is bounded.*

In particular, a subset of a ϕ -Šerstnev PN space that is a TV space is \mathcal{D} -bounded if and only if it is bounded.

PROOF. Suppose that (a) holds. For every $n \in \mathbb{N}$, there is a $k \in \mathbb{N}$ such that $\nu_{p/k}(1/n) > 1 - 1/n$ for all $p \in A$. Since ϕ is non-decreasing and continuous at infinity, there exists an $x_0 \in \mathbb{R}^+$ such that for all $x \geq x_0$, $\phi^\wedge(\phi(x)/k) \geq 1/n$. Thus, for every $p \in A$ and $x \geq x_0$, we obtain

$$\nu_p(x) = \nu_{k\frac{p}{k}}(x) = \nu_{\frac{p}{k}}\left(\phi^\wedge\left(\frac{\phi(x)}{k}\right)\right) \geq \nu_{\frac{p}{k}}\left(\frac{1}{n}\right) > 1 - \frac{1}{n},$$

so that, $R_A(x) \geq 1 - 1/n$, i.e. $R_A \in \mathcal{D}^+$ as desired.

Conversely, suppose that A is \mathcal{D} -bounded. Then, for every $n \geq 1$ there is an $x_n > 0$ such that $R_A(x_n) > 1 - 1/n$. Thus, $\nu_p(x_n) \geq R_A(x_n) > 1 - 1/n$, for all $p \in A$. As before, there exists a $k \in \mathbb{N}$ such that $\phi^\wedge(k\phi(1/n)) \geq x_n$. Then, for all $p \in A$,

$$\nu_{\frac{p}{k}}\left(\frac{1}{n}\right) = \nu_p\left(\phi^\wedge\left(k\phi\left(\frac{1}{n}\right)\right)\right) \geq \nu_p(x_n) > 1 - \frac{1}{n},$$

as desired. Finally, (a) implies (c) because $kN_\theta(1/n)$ is contained in $N_\theta(1/n)^{[k]}$. \square

Example 8.4 If $(V, \|\cdot\|, G)$ is a simple space with $G \notin \mathcal{D}^+$, then it is a Šerstnev PN space and its topology is discrete, thus not a TV space. In this case, a single set $\{p\}$, with $p \in V \setminus \{\theta\}$, is bounded but not \mathcal{D} -bounded.

9 Boundedness in F -normable PN spaces

We include a section treating the following unsolved problem:

Problem 9.1 Determine the class of all TV PN spaces where \mathcal{D} -bounded and bounded subsets coincide.

Proposition 9.2 *Such spaces satisfy that $\nu(V) \subseteq \mathcal{D}^+$.*

PROOF. Indeed, for every $p \in V$, the map $\mathbb{R} \rightarrow V$, given by $\lambda \mapsto \lambda p$, is continuous. This implies $\{p\}$ bounded. Hence, $\nu_p \in \mathcal{D}^+$. \square

However, the condition $\nu(V) \subseteq \mathcal{D}^+$ is not sufficient to have the equivalence between boundedness and \mathcal{D} -boundedness (see example below).

Proposition 9.3 *Let (V, g) be an F -normed space and (V, ν, τ, τ^*) be any pre-PN space with $\nu_p = \varepsilon_{g(p)}$. Then A is \mathcal{D} -bounded if and only if $g(A)$ is bounded in \mathbb{R}^+ .*

PROOF. Suppose that A is not \mathcal{D} -bounded, then $\lim_{x \rightarrow \infty} R_A(x) \neq 1$, hence this limit must be 0. Hence, for every $k \geq 1$ there exists a $p_k \in A$ such that $\varepsilon_{g(p_k)}(k) = 0$. This implies $g(p_k) \geq k$ for all $k \geq 1$, and therefore $g(A)$ is unbounded. The converse can be proved similarly. \square

Proposition 9.4 *Let (V, g) be an F -normed space. Then:*

1. *If A is \mathcal{D} -bounded, so is kA for all $k \in \mathbb{R}^+$.*
2. *Every bounded subset is \mathcal{D} -bounded.*

PROOF. For the first part, suppose that kA is not \mathcal{D} -bounded, for a natural number k . Then, there exists a sequence $(kp_r) \subseteq kA$ with $p_r \in A$ and $g(kp_r)$ converging to infinity. Since, $g(kp_r) \leq g(p_r) + g((k-1)p_r)$, we have that either $g(p_r)$ or $g((k-1)p_r)$ tends to infinity. By induction we can obtain that $g(p_r)$ tends to infinity.

For the second part, let A be a bounded subset of V . Suppose that A is not \mathcal{D} -bounded. Then, there exists a sequence $(p_r) \subseteq A$ with $g(p_r)$ converging to infinity. Since (p_r) is bounded, given $n = 1$ there exists a k such that $g(p_r/k) < 1$ for all $r \geq 1$. But by part 1, $g(p_r/k)$ converges to infinity, which is a contradiction. \square

Example 9.5 Consider the PN space $(\mathbb{R}, \nu, \tau_M, \mathbf{M})$ (see Theorem 5 and Example 4 in [10]) where $\nu_p = \varepsilon_{\|p\|/(1+\|p\|)}$. Then $\nu_p \geq \varepsilon_1 \in \mathcal{D}^+$ for all $p \in \mathbb{R}$. Thus \mathbb{R} is \mathcal{D} -bounded, but of course it is not bounded.

Another open problem related to problem 1.1 is the following:

Problem 9.6 Determine the class of all PN spaces (V, ν, τ, τ^*) , with τ^* Archimedean (thus TV spaces), where \mathcal{D} -bounded and bounded subsets coincide.

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